FALL 2025 MATH 147 : MIDTERM EXAM 2 SOLUTIONS

Please work each problem on a separate sheet of paper, using the reverse side if necessary. Be sure to put your name on each page of your solutions.

You must show all work to receive full credit. Good luck on the exam!

True-False. 5 points each

(i) Let B denote the solid box in $\mathbb{R}^3$ with vertices $(a,0,0),(a,b,0),(0,b,0),(a,0,c),(0,0,c),(0,b,c),(a,b,c)$. Then $\int\int\int xyz\ dV = \int_0^a\int_0^b\int_0^c xyz\ dx\ dy\ dz$. False. dx and dz are interchanged.

(ii) Suppose we have to maximize the function $f(x,y,z)$ with constraints $g(x,y,z) = h(x,y,z) = k(x,y,z) = 0$. Then we must solve the system of equations: $\nabla f = \lambda_1\nabla g + \lambda_2\nabla h + \lambda_3\nabla k$ together with $g(x,y,z) = 0, h(x,y,z) = 0, k(x,y,z) = 0$. True.

(iii) When approximating $\int\int_R f(x,y)\ dA$ using the xy coordinate system, we must cover R with increasingly small rectangles. False. One may cover R with any shapes of increasingly small areas.

(iv) $\int_{-\infty}^{\infty} e^{-x^2}\ dx = 2\pi$. False. The correct value is $\sqrt{\pi}$.

Short Answer. 10 points each

(i) For the region D bounded by the lines $y = 0, y = \frac{x}{2}$ and $x + y = 1$, find a change of variables that makes it possible to calculate $\int\int_D \sqrt{\frac{x+y}{x-2y}}\ dA$, and then set up the resulting double integral. Do not calculate the double integral.

Solution. We first note that D is the triangle in the xy-plane with vertices (0,0), (1, 0), $(\frac{2}{3},\frac{1}{3})$. We want to have $u = x + y$ and $v = x - 2y$, which will enable us to antidifferentiate the integrand $\sqrt{\frac{u}{v}}$, so we think of these equations as as the coordinates of $F(x,y)$, the inverse of our change of variables transformation. Solving for x, y in terms of u, v gives $x = \frac{2u+v}{3}$ and $y = \frac{u-v}{3}$. Thus, we take $G(u,v) = (\frac{2u+v}{3},\frac{u-v}{3})$. Since $x + y = 1$ is one edge of D and $u = x + y$, F transforms this line to $u = 1$. Similarly, since $v = x - 2y$, F transforms the line $y = \frac{x}{2}$ to the line $v = 0$. Finally, when $y = 0$, $u = x = v$, so that F transforms the line $y = 0$ to the line $v = u$. It follows that F transforms D to the triangle in the uv-plane having vertices (0,0), (1,0), (1,1). Since the absolute value of the Jacobian of G is easily seen to be $\frac{1}{3}$, it follows that

$$\int\int_D \sqrt{\frac{x+y}{x-2y}}\ dA = \int_0^1\int_0^u \sqrt{\frac{u}{v}}\ \frac{1}{3}\ dv\ du.$$

Note that the inner integral is an improper integral, but a convergent proper integral, so it is easy to obtain a final answer.

(ii) Verify that $F(x,y) = (2x - 5y, -x + 3y)$ is the inverse transformation of $G(u,v) = (3u + 5v, u + 2v)$.

Solution. Substituting $G(u,v)$ into $F(x,y)$ we get

$$\begin{aligned} F(G(u,v)) &= F(3u+5v, u+2v) \\ &= (2(3u+5v) - 5(u+2v), -(3u+5v) + 3(u+2v)) \\ &= (6u + 10v - 5u - 10v, -3u - 5v + 3u + 6v) \\ &= (u,v) \end{aligned}$$

Substituting $F(x,y)$ into $G(u,v)$ we get

$$\begin{aligned} G(f(x,y)) &= G(2x-5y, -x+3y) \\ &= 3(2x-5y) + 5(-x+3y), 2x - 5y + 2(-x+3y)) \\ &= 6x - 15y - 5x + 15y, 2x - 5y - 2x + 6y) \\ &= (x,y). \end{aligned}$$

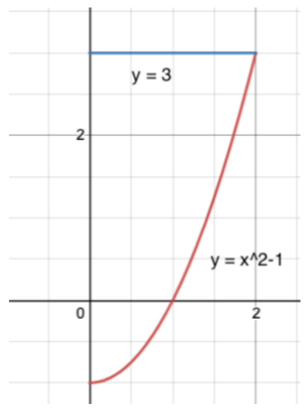
(iii) Let R denote the the set of points in $\mathbb{R}^2$ satisfying $0 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$. Use a transformation from the (r,θ)-plane to rewrite $\int\int_R xy\ dA$ as an iterated double integral in r, θ. You do not have to calculate the resulting double integral.

Solution. We let $G(r,\theta) = (ar\cos(\theta), br\sin(\theta))$ with $0 \le \theta \le 2\pi$ and $0 \le r \le 1$.. The Jacobian of this transformation is $\det\begin{pmatrix} a\cos(\theta) & -ar\sin(\theta) \\ b\sin(\theta) & rb\cos(\theta)\end{pmatrix} = abr$. Thus,

$$\int\int_R xy\ dA = \int_0^{2\pi}\int_0^2 (ar\cos(\theta))(br\sin(\theta)\cdot abr\ dr\ d\theta = \int_0^{2\pi}\int_0^1 a^2b^2\cos(\theta)\sin(\theta)r^3\ dr\ d\theta.$$

(iv) Explain how you would calculate $\int_0^2\int_{x^2-1}^3 e^{(y+1)^{\frac{3}{2}}}\ dy\ dx$ and why this works, though you do not have to fully calculate the double integral.

Solution. As given, we cannot antidifferentiate the integrand with respect to y. Since the given integral is $\int\int_D e^{(y+1)^{\frac{3}{2}}}\ dA$, for D



by Fubini's theorem we can change the order of integration, thereby getting $\int_0^3\int_0^{\sqrt{y+1}} e^{(y+1)^{\frac{3}{2}}}\ dx\ dy$. When we integrate with respect to x we get

$$\int_0^2 \{xe^{(y+1)^{\frac{3}{2}}}\}\Big|_{x=0}^{x=\sqrt{y+1}}\ dy = \int_0^3 \sqrt{y+1}e^{(y+1)^{\frac{3}{2}}}\ dy,$$

which can easily be solved by u-substitution.

(v) Set up $\int\int\int_B xy - z^2\ dV$ as an iterated integral, where B is the region in $\mathbb{R}^3$ between the planes $z = x$ and $z = y$ that lies over the square $D : [-1,1]\times[-1,1]$. You do not have to solve the triple integral.. Hint: You may have to write this as the sum of two integrals.

Solution. We first note that for points (x,y) in the xy-plane, the plane $z = y$ lies above the plane $z = x$, if (x,y) lies above the line $y = x$, and the plane $z = x$ lies above the plane $z = y$, if (x,y) lies below the line $y = x$. Thus,

$$\int\int\int_B xy - z^2\ dV = \int_{-1}^1\int_x^1\int_x^y xy - z^2\ dz\ dy\ dx + \int_{-1}^1\int_{-1}^x\int_y^x xy - z^2\ dz\ dy\ dx.$$

Long Answer. 20 points each

1. Use a Lagrange multiplier to find the maximum volume of a box inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Hint: Assume the box is centered at the origin and that the sides of the box are parallel to the coordinate planes.

Solution. Since the box is centered at (0,0,0) we can think of the vertices as being $(\pm x, \pm y, \pm z)$ and these points are on the ellipse. The lengths of the sides are $2x, 2y, 2z$ so the volume of the box is $V = 8xyz$. The constraint equation is $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. Setting $\nabla V = \lambda\nabla g$ we get

$$8yz = \lambda(\frac{2x}{a^2})$$
$$8xz = \lambda(\frac{2y}{b^2})$$
$$8xy = \lambda(\frac{2z}{c^2})$$

Multiplying the first equation by x, the second equation by y and the third equation by z we see that $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$. Using this in the constraint equation gives $3 \cdot \frac{x^2}{a^2}$, so $x = \frac{a}{\sqrt{3}}$. Similarly $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$. Thus the maximum volum is $V = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$.

2. Calculate $\int\int\int_B x + y + z \; dV$, where B is the solid whose lower half is the solid hemisphere of radius 2 centered at the origin, below the xy-plane and whose upper half is the inverted solid cone with radius 2 and height equals 4. This looks like half of a solid ball with a pointy witches hat. You can use the formula $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$.

Solution. If we let B_1 denote the hemisphere and B_2 denote the inverted cone, then we seek

$$\int\int\int_{B_1} x + y + z \; dV + \int\int\int_{B_2} x + y + z \; dV.$$

Using spherical coordinates, we have $\int\int\int_{B_1} x + y + z \; dV =$

$$\begin{aligned}
&= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_0^2 \{\rho\cos(\theta)\sin(\phi) + \rho\sin(\theta)\sin(\phi) + \rho\cos(\phi)\} \cdot \rho^2 \sin(\phi) \; d\rho \; d\phi \; d\theta \\
&= \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_0^2 \rho^3\{\cos(\theta)\sin(\phi) + \sin(\theta)\sin(\phi) + \cos(\phi)\} \cdot \sin(\phi) \; d\rho \; d\phi \; d\theta \\
&= \frac{2^4}{4} \cdot \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \{\cos(\theta)\sin(\phi) + \sin(\theta)\sin(\phi) + \cos(\phi)\} \cdot \sin(\phi) \; d\phi \; d\theta \\
&= 4 \cdot \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \{\cos(\theta) + \sin(\theta)\}\sin^2(\phi) + \cos(\phi)\sin(\phi) \; d\theta \; d\phi \\
&= 4 \cdot \int_{\frac{\pi}{2}}^{\pi} 2\pi\cos(\phi)\sin(\phi) \; d\phi, \quad \text{since} \int_0^{2\pi} \cos(\theta) + \sin(\theta) \; d\theta = 0 \\
&= 8\pi \cdot \frac{1}{2}\sin^2(\phi)\}_{\phi=\frac{\pi}{2}}^{\phi=\pi} \\
&= -4\pi
\end{aligned}$$

For the second integral we note that the equation of the boundary of the inverted cylinder is

$$z = -\frac{4}{2}\sqrt{x^2 + y^2} + 4 = -2\sqrt{x^2 + y^2} + 4.$$

Letting D denote the solid disk of radius 2 in the xy-plane centered at (0,0), we have $\int\int\int_B x + y + x \; dV =$

$$\begin{aligned}
&= \int\int_D \{\int_0^{-2\sqrt{x^2+y^2}+4} x + y + z \; dz\} \; dA \\
&= \int\int_D \{(x + y)z + \frac{z^2}{2}\}\Big|_{z=0}^{z=-2\sqrt{x^2+y^2}+4} \; dA \\
&= \int\int_D \{(x + y)(-2\sqrt{x^2 + y^2} + 4) + \frac{(-2\sqrt{x^2 + y^2} + 4)^2}{2}\} \; dA \\
&= \int_0^2\int_0^{2\pi} \{(r\cos(\theta) + r\sin(\theta)(-2r + 4) + \frac{1}{2}(4r^2 - 16r + 16)\} \cdot r \; d\theta \; dr \\
&= \pi\int_0^2 4r^3 - 16r^2 + 16r \; dr, \quad \text{since} \int_0^{2\pi} \cos(\theta) + \sin(\theta) \; d\theta = 0 \\
&= \pi \cdot \{16 - \frac{128}{3} + 32\} = \frac{16\pi}{3}.
\end{aligned}$$

Therefore, $\int\int\int_B x + y + z \; dV = \frac{16\pi}{3} - 4\pi = \frac{4\pi}{3}$.

3. Let R denote the parallelogram in the xy-plane having vertices $(-2, 0), (0, -2), (2, 0), (0, 2)$.

(a) Find a transformation $G(u, v)$ from the uv-plane to the xy-plane that takes the unit square U in the uv-plane to R. You must give details to explain why $G(U) = R$.

(b) Use your transformation from (a) to calculate $\int\int_R 3x + y^2 \; dA$.

Solution. For part (a), the transformation $G(u, v)$ can be thought of a linear transformation followed by a translation of the parallelogram P with vertices (0,0), (2, -2), (4, 0), (2,2) two units to the left. The linear transformation taking U to P is given by $H(u, v) = (2u + 2v, -2u + 2v)$. Thus, if we follow this by a translation, then $G(u, v) = (-2 + 2u + 2v, -2u + 2v)$. Note that $H(u, v)$ takes the e_1 vector in the uv-plane to the vector (2, -2) in the xy-plane and $H(u, v)$ takes e_2 in the uv-plane vector (2,2) in the xy-plane. If (a, b) is a point in U, then $H(a, b) = (2a + 2v, -2a + 2b) = a(2, -2) + b(2, 2)$, is a point in P,. Thus, $H(u, v)$ takes U to P. Since $G(u, v)$ is just $H(u, v)$ followed by a translation two units to the left, we have $G(U) = R$.

For part (b), the Jacobian of G is easily seen to be 8. Thus,

$$\begin{aligned}\int\int_R 3x + y^2\ dA &= \int_0^1\int_0^1 \{3(2u + 2v - 2) + (-2u + 2v)^2\} \cdot 8\ du\ dv \\ &= 8 \cdot \int_0^1\int_0^1 4u^2 - 8uv + 4v^2 + 6u + 6v - 6\ du\ dv \\ &= 8 \cdot \int_1^1 \frac{4}{3} - 4v + 4v^2 + 3 + 6v - 6\ dv \\ &= 8 \cdot \frac{2}{3} = \frac{16}{3}.\end{aligned}$$

4. Use what you know about improper single and double integrals to calculate $\int\int\int_B e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV$, where B denotes the set of points in $\mathbb{R}^3$ whose distance from the origin is greater than or equal to 1.

Solution. Let $R > 1$ and C_R denote the region in $\mathbb{R}^3$ between the sphere of radius R centered at the origin and the sphere of radius 1 centered at the origin. Then

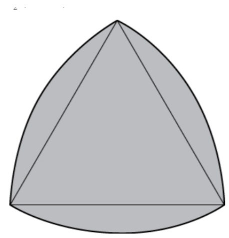
$$\int\int\int_B e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV = \lim_{R\to\infty}\int\int\int_{C_R} e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV.$$

Using spherical coordinates we have

$$\begin{aligned}\int\int\int_{C_R} e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV &= \int_0^{2\pi}\int_0^{\pi}\int_1^R e^{-(\rho^2)^{\frac{3}{2}}}\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta \\ &= 2\pi\int_0^{\pi}\int_1^R e^{-\rho^3}\rho^2\sin(\phi)\ d\rho\ d\phi \\ &= 4\pi\int_1^R e^{-\rho^3}\rho^2\ d\phi,\ \text{since}\ \int_0^{\pi}\sin(\phi)\ d\phi = 2 \\ &= 4\pi\{-\frac{1}{3}e^{-\rho^3}\}_1^R \\ &= 4\pi\{-\frac{1}{3}e^{-R^3} + \frac{1}{3e}\}.\end{aligned}$$

Takiing the limit as $R \to \infty$ of this last expression gives $\int\int\int_B e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}\ dV = \frac{4\pi}{3e}$.

Bonus Problem. The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$. (15 points)



Solution. If we think of the lower left corner of the inner equilateral triangle as being at the origin, then the area of the first pie-shaped region is $\int_0^{\frac{\pi}{3}}\int_0^s r\ drd\theta = \frac{\pi s^2}{6}$. The area of three such regions is $\frac{\pi s^2}{2}$. These three regions cover the Reuleaux triangle, but in doing so, we have counted the area of the inner equilateral triangle three times. Since the area of the triangle is $\frac{\sqrt{3}s^2}{4}$, the area of the Reuleaux triangle is:

$$\frac{\pi s^2}{2} - 2 \cdot \frac{\sqrt{3}s^2}{4} = \frac{s^2}{2}(\pi - \sqrt{3}).$$